

Comments on M_{24} representations and CY_3 geometries

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ABSTRACT: We show using string dualities that Mathieu moonshine controls Gromov-Witten invariants *and* periods of the holomorphic 3-form Ω for certain CY_3 manifolds. We also discuss how the period vectors appear in flux compactifications on these CY_3 manifolds and work out the connection between the sporadic group M_{24} and the Yukawa couplings in four dimensional theories that arise from heterotic string theory compactifications on these CY_3 manifolds.

1 Introduction

In 2010 Eguchi, Ooguri and Tachikawa [1] showed that the elliptic genus of the $K3$ manifold can be expanded in such a way that the first few expansion coefficients are sums of dimensions of irreducible representations of the largest Mathieu group M_{24} . This connection between the elliptic genus of $K3$ and M_{24} was checked and confirmed in [2–5].¹ In 2012, Gannon proved [10] that all the expansion coefficients appearing in the elliptic genus are sums of irreducible representations of M_{24} . Despite all this work, there are still many interesting questions related to this ‘Mathieu moonshine’ that have not yet been answered. For example, no $\mathcal{N} = (4, 4)$ non-linear sigma model with $K3$ target has M_{24} as its symmetry group [11]. So why does the elliptic genus of $K3$ exhibit this connection to M_{24} ? One possible explanation, currently pursued in, for example, [12, 13], is that the symmetries of different points in $K3$ moduli space combine to give M_{24} . An alternative idea is that models which preserve only $\mathcal{N} = (0, 4)$ worldsheet supersymmetry and that are connected to $\mathcal{N} = (4, 4)$ non-linear sigma model with $K3$ target, have as their symmetry group the full M_{24} group [14, 15].

Since Mathieu moonshine involves the $K3$ manifold that has played a major role in compactifications of superstring theories and in string dualities, it is very interesting for string theorists. We are currently in the process of understanding the implications of this moonshine phenomenon for superstring compactifications and have already obtained a variety of new insights: For example, it was shown in [16] that certain one-loop amplitudes in compactifications of type II string theory on $K3 \times T^2$ are related to the elliptic genus of $K3$ and therefore to Mathieu moonshine. In [17], the authors found that certain BPS states in type II string theory compactified on $S^1 \times K3$ are related to a particular mock modular form that is closely related to the elliptic genus of $K3$. Compactifying the heterotic string theory on $K3 \times T^2$, the authors of [14] showed that the sums of irreducible representations of M_{24} that appear in Mathieu moonshine also appear (albeit in a less direct manner) in the prepotential of the resulting four dimensional $\mathcal{N} = 2$ theories.² To support the conjecture that Mathieu moonshine plays a role in these $\mathcal{N} = 2$ compactifications, a variety of twined elliptic genera (i.e. the analogue of the McKay-Thompson series for the Monster) were calculated in [15], in which the authors twined by explicit symmetries of heterotic GLSMs with $K3$ target, for various instanton embeddings. For some of these symmetries, the twined elliptic genera reproduced the graded traces predicted by Mathieu moonshine. These heterotic theories are dual to type IIA compactifications on CY_3 manifolds X_n that are elliptic fibrations over the Hirzebruch surfaces \mathbb{F}_n for $n = 0, 1, \dots, 12$. In these dual type IIA theories the prepotential receives instanton corrections and those are by duality related to the Mathieu group M_{24} [14].

¹For very interesting generalizations of this moonshine see [6–8] and [9].

²In the case of the standard embedding, where there exists a $(4, 4)$ locus in the $(0, 4)$ moduli space, it is perhaps reasonable to decompose the prepotential into $\mathcal{N} = 4$ characters to observe the appearance of M_{24} representations. This would correspond on the type IIA side to a compactification on the threefold with base \mathbb{F}_{12} . However, it is unclear why the $\mathcal{N} = 4$ characters, rather than e.g. Virasoro characters augmented by a $U(1)$ current algebra, continue to work for other embeddings. It would be interesting to understand this point better; for now, we can simply say that the $\mathcal{N} = 4$ decompositions, perhaps miraculously, work.

More specifically, the instanton corrections are determined by the Gromov-Witten invariants of the CY_3 manifolds X_n and these are connected to Mathieu moonshine. This extends the usual connection between number theory and representation theory that is heralded by the appearance of moonshine to also include (algebraic) geometry. Furthermore, the corrections to the prepotential determine the gauge couplings in the four-dimensional $\mathcal{N} = 2$ spacetime theories. Hence, the 1-loop corrections to the gauge couplings are implicated in Mathieu moonshine. Such a connection appears more generally in heterotic string theory compactifications. It was shown in [18] that for almost all four-dimensional $\mathcal{N} = 1$ theories that arise from heterotic orbifold compactifications, the gauge kinetic functions (and therefore the gauge couplings) receive a universal one-loop correction that is connected to the Mathieu group M_{24} .

We see that Mathieu moonshine has already lead to a variety of intriguing new insights for several different compactifications of superstring theories. In this paper we add to this list by applying mirror symmetry to the above type IIA compactifications on the CY_3 manifolds X_n that are elliptic fibrations over the Hirzebruch surfaces \mathbb{F}_n . Mirror symmetry relates the Gromov-Witten invariants of X_n to the periods of the holomorphic 3-form Ω of the mirror Y_n . We explicitly work out the connection between these periods and representations of M_{24} for Y_n with $n = 0, 1, 2$, though the results will generalize to all n in an obvious way. Having implicated the holomorphic 3-forms of the Y_n in Mathieu moonshine, we note that for $n = 2, 4, 6, 12$, the X_n are given as hypersurfaces in the weighted projective space $\mathbb{WP}_{1,1,n,2n+4,3n+6}$ and the mirror manifolds Y_n can be obtained from a Greene-Plesser type construction [19]. This means that one expects that a subspace of the complex structure moduli space of these particular X_n is the same as the complex structure moduli space of the Y_n (and likewise for the quantum Kähler moduli space). For at least $n = 2, 4, 6, 12$ there would then be a connection between M_{24} and the Kähler as well as the complex structure sector of the X_n and Y_n . Having established such a link, we then proceed and discuss two implications for physically interesting theories. First we study flux compactifications on X_n and Y_n and show how M_{24} representations appear in the Gukov-Vafa-Witten superpotential. Then we discuss compactification of the heterotic $E_8 \times E_8$ string theory on X_n and Y_n and find that the Yukawa couplings and therefore the masses of the particles in the resulting four-dimensional $\mathcal{N} = 1$ theories are implicated in Mathieu moonshine as well.

The outline of the paper is as follows: In section 2, we review Mathieu moonshine and show how through string dualities it controls Gromov-Witten invariants *or* periods of the holomorphic 3-form Ω for certain CY_3 manifolds. Then we argue in section 3 that at least for some CY_3 manifolds the complex structure *and* Kähler moduli space is implicated in Mathieu moonshine. Next we study flux compactifications on these manifolds in section 4 and explicitly show how M_{24} representations appear in the superpotential. In section 5 we show for certain compactifications of the heterotic string theory, how the Yukawa couplings of the 4d $\mathcal{N} = 1$ theories are related to M_{24} . We summarize our findings and point out interesting future directions in section 6. Appendix A provides a concise introduction to mirror symmetry and appendix B lists topological data for three CY_3 manifolds that are of

particular interest to us.

2 Mathieu Moonshine and the holomorphic 3-form Ω

In this section we first review Mathieu moonshine that was discovered in [1]. There the authors expand the elliptic genus of the $K3$ manifold and find that the expansion coefficients are sums of dimensions of irreducible representations of the largest Mathieu group M_{24} . Then we use the duality between heterotic string theory compactifications on $K3 \times T^2$ and type IIA compactifications on CY_3 manifolds X_n that are elliptic fibrations over \mathbb{F}_n to discuss (following [14]) how Mathieu moonshine is connected to the Gromov-Witten invariants of the X_n . Using mirror symmetry we finally connect Mathieu moonshine to the holomorphic 3-form Ω of Y_n , that are the mirror CY_3 manifolds of the X_n . We then argue using the Greene-Plesser construction of mirror pairs that at least some of X_n and Y_n exhibit a connection between M_{24} and *both* their Gromov-Witten invariants and their holomorphic 3-form Ω .

2.1 Mathieu moonshine

The elliptic genus is defined as the following trace over the RR sector of an $\mathcal{N} = (2, 2)$ superconformal field theory

$$\mathcal{Z}_{elliptic}(q, y) = \text{Tr}_{RR} \left((-1)^{F_L + F_R} q^{L_0 - \frac{c}{24}} y^{J_0} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right). \quad (2.1)$$

Here $F_{L/R}$ denotes the left/right moving fermion number and y is a chemical potential for the left-moving $U(1)$ charge measured by J_0 . Since only the right-moving Witten index $(-1)^{F_R} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}$ appears in $\mathcal{Z}_{elliptic}$, it does not depend on \bar{q} . For the particular case of $K3$, the elliptic genus was calculated in 1989 in [20]. It wasn't until 2010, however, that Eguchi, Ooguri and Tachikawa [1] noticed that the coefficients appearing in the $K3$ elliptic genus expanded in terms of $\mathcal{N} = 4$ characters are related to the dimensions of irreducible representations of M_{24} . In particular, if we define the $\mathcal{N} = 4$ superconformal characters [20] (please see appendix A of [14] for our conventions for the Jacobi θ -functions)

$$\text{ch}_{h=\frac{1}{4}, l=0}(q, y) = -\frac{iy^{\frac{1}{2}}\theta_1(q, y)}{\eta(q)^3} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n+1)} y^n}{1 - y q^n}, \quad (2.2)$$

$$\text{ch}_{h=n+\frac{1}{4}, l=\frac{1}{2}}(q, y) = q^{n-\frac{1}{8}} \frac{\theta_1(q, y)^2}{\eta(q)^3}, \quad (2.3)$$

then one finds the following expansion [1]

$$\mathcal{Z}_{elliptic}^{K3}(q, y) = 8 \left[\left(\frac{\theta_2(q, y)}{\theta_2(q, 1)} \right)^2 + \left(\frac{\theta_3(q, y)}{\theta_3(q, 1)} \right)^2 + \left(\frac{\theta_4(q, y)}{\theta_4(q, 1)} \right)^2 \right] \quad (2.4)$$

$$= 24 \text{ch}_{h=\frac{1}{4}, l=0}(q, y) + \sum_{n=0}^{\infty} A_n \text{ch}_{h=n+\frac{1}{4}, l=\frac{1}{2}}(q, y). \quad (2.5)$$

The $24 = 23+1$ as well as the first few A_n where identified in [1] as sums of irreducible representations of M_{24}

$$\begin{aligned}
A_0 &= -2 = -1 - 1, \\
A_1 &= 90 = 45 + \overline{45}, \\
A_2 &= 462 = 231 + \overline{231}, \\
A_3 &= 1540 = 770 + \overline{770}, \\
A_4 &= 4554 = 2277 + \overline{2277}, \\
&\dots
\end{aligned} \tag{2.6}$$

It was proven in [10] that all the A_n for $n \geq 1$ are sums of irreducible representations of M_{24} with only positive coefficients.

2.2 M_{24} in Type II $\mathcal{N} = 2$ theories

This connection between the elliptic genus of $K3$ and the Mathieu group M_{24} is still not understood and one might hope that studying the appearance of this Mathieu moonshine in different string theory settings might help understand it better. In addition, this might lead to new insights in otherwise well-understood string compactifications and connections between physical observables and the Mathieu group M_{24} in certain toy models. Of particular interest to us is [14], where it was shown that the elliptic genus of $K3$ appears in compactifications of the heterotic string theory and that, by duality, the Gromov-Witten invariants of certain CY_3 manifolds are related to the Mathieu group M_{24} . After quickly reviewing these results we will extend them and show explicitly how the holomorphic three form of certain CY_3 manifolds is related to M_{24} .

The heterotic $E_8 \times E_8$ string theory compactified on $K3 \times T^2$ leads to a four dimensional spacetime theory with $\mathcal{N} = 2$ supersymmetry (see for example [21] for a nice review of basic facts about these theories). In order to satisfy the Bianchi identity for the H_3 field one has to turn on a non-trivial gauge bundle inside one or both of the E_8 gauge groups. In particular, in the absence of NS5-branes, we have to embed a total of 24 instantons into the two E_8 's which leads to 13 different cases due to the symmetry that exchanges the two E_8 's. We embed $(12 - n, 12 + n)$ instantons in $E_8 \times E_8$ and take w.l.o.g. $n = 0, 1, \dots, 12$. These 13 cases are perturbatively inequivalent, however, each case can be further subdivided based on the particular subgroup $G \times G' \subset E_8 \times E_8$ in which one turns on the instantons.

For $n = 0, 1, 2$ the instantons generically break the $E_8 \times E_8$ gauge symmetry and there are only three vector multiplets whose scalar components we denote by S, T and U . S is the axio-dilaton, while T and U control the size and complex structure of the two torus T^2 . For $n > 2$ there are additional Wilson line moduli V^i . As was shown in [14], after setting the Wilson line moduli to zero $V^i = 0$, the prepotential for the thirteen four dimensional $\mathcal{N} = 2$ spacetime theories is always the same and is directly related to the elliptic genus of $K3$ and

therefore to M_{24} .³ In particular (up to a quadratic polynomial in S , T and U) it is given by

$$F = STU + \frac{1}{3}U^3 + \frac{1}{(2\pi i)^3}c(0)\zeta(3) - \frac{2}{(2\pi i)^3} \sum_{\substack{k>0, l \in \mathbb{Z} \\ k=0, l>0}} c(kl) Li_3 \left(q_T^k q_U^l \right) + \mathcal{O}(e^{2\pi i S}), \quad (2.8)$$

where $\zeta(3) \approx 1.2$ is the Riemann zeta function, $q_U = e^{2\pi i U}$, $q_T = e^{2\pi i T}$, the polylogarithm is defined as $Li_p = \sum_{n=1}^{\infty} \frac{x^n}{n^p}$ and the coefficients $c(m)$ are obtained from the expansion

$$\frac{E_4(q)E_6(q)}{\eta(q)^{24}} = \sum_{m \geq -1} c(m)q^m = \frac{1}{q} - 240 - 141444q - \dots, \quad \text{and} \quad c(m) = 0 \quad \forall m < -1, \quad (2.9)$$

where $E_i(q)$ are the Eisenstein series (see appendix A in [14] for a definition).

From the explicit derivation of the prepotential one finds that $E_6(q)$ and therefore the $c(m)$ in the prepotential (2.8) are related to the elliptic genus of $K3$. Explicitly one has

$$\begin{aligned} -\frac{4E_6(q)}{\eta(q)^{12}} &= \left(\frac{\theta_2(q)}{\eta(q)} \right)^6 \mathcal{Z}_{elliptic}^{K3}(q, -1) + \left(\frac{\theta_3(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} \mathcal{Z}_{elliptic}^{K3}(q, -q^{\frac{1}{2}}) \\ &\quad - \left(\frac{\theta_4(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} \mathcal{Z}_{elliptic}^{K3}(q, q^{\frac{1}{2}}) \\ &= 24g_{h=\frac{1}{4}, l=0}(q) + g_{h=\frac{1}{4}, l=\frac{1}{2}}(q) \sum_{n=0}^{\infty} A_n q^n, \end{aligned} \quad (2.10)$$

where the $24=23+1$ and the A_n 's can be decomposed into irreps of M_{24} as in (2.6) and we defined

$$\begin{aligned} g_{h=\frac{1}{4}, l}(q) &= \left(\frac{\theta_2(q)}{\eta(q)} \right)^6 \text{ch}_{h=\frac{1}{4}, l}(q, -1) + \left(\frac{\theta_3(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} \text{ch}_{h=\frac{1}{4}, l}(q, -q^{\frac{1}{2}}) \\ &\quad - \left(\frac{\theta_4(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} \text{ch}_{h=\frac{1}{4}, l}(q, q^{\frac{1}{2}}). \end{aligned} \quad (2.11)$$

Having established this connection between the Mathieu group M_{24} and the $\mathcal{N} = 2$ prepotential in the spacetime theory, the authors of [14] used the fact that these compactifications of the heterotic $E_8 \times E_8$ string theory are dual to compactifications of type IIA on CY_3 manifolds X_n that are elliptic fibrations over the Hirzebruch surfaces \mathbb{F}_n , where again $n = 0, 1, \dots, 12$.⁴ In the dual type IIA compactification the infinite sum in the prepotential (2.8) arises from instanton corrections and the $c(m)$ are related to the Gromov-Witten invariants of the CY_3 manifolds X_n . The prepotential on the type IIA side was recently calculated in [22, 23] for X_0, X_1 and X_2 and it matches the heterotic result (2.8) to leading order in q_T [14]. Thus,

³If one embeds all instantons in one E_8 and allows for non-zero Wilson lines for the other E_8 , then there is still a direct connection between the prepotential and M_{24} [14].

⁴We group together all the CY_3 manifolds that are dual to heterotic constructions with the same instanton numbers and collectively call them X_n . All manifolds for a given n are related by geometric transitions that correspond to (un-)higgsing the gauge group on the dual heterotic side.

there is a connection between Gromov-Witten invariants of certain CY_3 manifolds and the sporadic group M_{24} .

We now review that by mirror symmetry this implies that for certain CY_3 manifolds the holomorphic 3-form Ω is likewise connected to the Mathieu group M_{24} . Mirror symmetry, as we review in appendix A, is a duality between compactifications of type IIA string theory on a Calabi-Yau manifold X_n and type IIB string theory on the mirror Calabi-Yau manifold Y_n . The moduli space of four dimensional $\mathcal{N} = 2$ theories (locally) factorizes into a hypermultiplet part and a vector multiplet part. In our particular compactifications of the heterotic and type IIA string theories the vector multiplets are connected to M_{24} . In compactifications of type IIA string theory the vector multiplets arise from the Kähler moduli sector, while for the dual type IIB string theory compactifications the vector multiplets arise from the complex structure sector. So we expect that the mirror CY_3 manifolds Y_n have a complex structure moduli space that is related to M_{24} .

In particular, as discussed in appendix A.3, we can integrate the holomorphic three form Ω of the CY_3 manifolds Y_n over a canonical homology basis such that

$$z^i = \int_{A^i} \Omega, \quad \mathcal{F}_i(z) = \int_{B_i} \Omega. \quad (2.12)$$

In the basis of forms dual to $\{A^i, B_i\}$, it is often convenient to expand the 3-form as $\Omega = z^i \alpha_i - \mathcal{F}_i(z) \beta^i$. As we will explain, the \mathcal{F}_i exhibit interesting dependence on M_{24} via their dependence on the holomorphic prepotential \mathcal{F} : $\mathcal{F}_i = \partial_{z^i} \mathcal{F}$.

The prepotential that controls the vector multiplet moduli space for type IIB compactifications on Y_n is given by $\mathcal{F} = \frac{1}{2} z^i \mathcal{F}_i(z)$, which is a function of the projective coordinates z^i . The periods themselves are solutions of the Picard-Fuchs equations, which can be determined with the classical intersection numbers of the mirror, X_n , as input. The mirror map can also be inferred from the solutions to the Picard-Fuchs equations in an expansion around $z^i = 0$. This large complex structure point is mirror to the large radius point of X_n , so applying the mirror map in an expansion around this point enables us to read off the Gromov-Witten invariants of X_n . Therefore, the period vector of Y_n is controlled entirely by some classical topological numbers plus the Gromov-Witten invariants of its mirror X_n .

The Gromov-Witten invariants come from the worldsheet instanton corrections to the Kähler moduli space of X_n , which must be small for our perturbative expansion to be valid. It is important to remember, though, that the complex structure moduli space of Y_n is classically exact and its periods are expressible in a simple closed form in the z^i coordinates.

The period vector can be expressed in terms of the prepotential as (see Appendix A for details):

$$\Pi = \begin{pmatrix} 1 \\ t^i \\ \frac{\partial}{\partial t^i} \frac{\mathcal{F}}{(z^0)^2} \\ 2 \frac{\mathcal{F}}{(z^0)^2} - t^i \frac{\partial}{\partial t^i} \frac{\mathcal{F}}{(z^0)^2} \end{pmatrix}, \quad (2.13)$$

where $t^i = z^i/z^0$ are the three moduli dual to S, T, U on the heterotic side. Since the Hirzebruch surfaces may be viewed as certain \mathbb{P}^1 fibrations over \mathbb{P}^1 , the t^i measure the volumes of the elliptic fiber and the two \mathbb{P}^1 s: $t^i = \int_{C_i} (B + iJ)$, where B is the NS-NS field and J is the Kähler form.⁵ Finally, we can write the period vector even more explicitly by plugging in $F := \mathcal{F}/(z^0)^2$

$$F = \left(\frac{\kappa_{ijk}^0}{6} t^i t^j t^k + \frac{1}{2} a_{ij} t^i t^j + b^i t^i + \frac{\chi(X_n) \zeta(3)}{2(2\pi i)^3} + \frac{1}{(2\pi i)^3} \sum_{(n_i)} N_{(n_i)} Li_3(q_i^{(n_i)}) \right). \quad (2.14)$$

The κ_{ijk}^0 are the classical triple intersection numbers of X_n . a_{ij} and b_i are also classical topological numbers which we define in Appendix A. We list their numerical values for X_0, X_1 and X_2 in Appendix B. The $N_{(n_i)}$ are the Gromov-Witten invariants of X_n , of which at least a subset is governed by Mathieu moonshine, as we will delineate shortly.

With the aforementioned substitution, the period vector becomes:

$$\Pi = \begin{pmatrix} 1 \\ t^i \\ \frac{\kappa_{ijk}^0}{2} t^j t^k + a_{ij} t^j + b^i + \partial_{t^i} (F_{inst}) \\ -\frac{\kappa_{ijk}^0}{6} t^i t^j t^k + b_i t^i + c + 2F_{inst} - t^i \partial_{t^i} F_{inst} \end{pmatrix}, \quad (2.15)$$

where we have defined $F_{inst} = \frac{1}{(2\pi i)^3} \sum_{(n_i)} N_{(n_i)} Li_3(q_i^{(n_i)})$. In practice, it is easiest to compute the prepotential (in the t^i coordinates, expanded around the large complex structure/large radius point) and the Gromov-Witten invariants directly from (2.15) or by computing a triple integral of $\kappa_{ijk}[X_n]$ (see Appendix A) with the classical topological numbers as input. This is what we have done; we record the $\kappa_{ijk}[X_n] = \bar{\kappa}_{ijk}[Y_n]$ for $n = 0, 1, 2$ to fifth order in the $q_i = e^{2\pi i t^i}$ in Appendix B.

Finally, we wish to verify that our mirror symmetry computations exhibit the moonshine that we expect from the heterotic/IIA duality described earlier. After computing the prepotential, we finally have all the necessary information in hand. First, we note that the duality is good on the heterotic side when the string coupling is small. This corresponds to $S \rightarrow 0$, which for us means “ignoring” instanton contributions from what in the notation of Appendix B we call q_2 in the elliptic fibration over \mathbb{F}_0 and q_3 for $\mathbb{F}_{1,2}$. We simply use the usual type IIA/heterotic dictionary [21] and match

$$-2c_{STU}(kl) = N_{II}(k+l, 0, k)[X_0] = N_{II}(k+l, k, 0)[X_{1,2}], \quad (2.16)$$

which are the coefficients of the F_{inst} on each side of the duality.⁶

⁵In the context of the type II string, we may view our compactification manifold as being either an elliptic fibration over \mathbb{F}_n or a K3 fibration over \mathbb{P}^1 . The elliptic fibration over \mathbb{F}_2 , which we will study extensively in the next section, is a hypersurface $X_{24}(1, 1, 2, 8, 12)$ in a weighted projective space. The K3 fiber of the latter point of view is a hypersurface in $X_{12}(1, 1, 4, 6)$.

⁶The notation $N_{II}(k+l, k, 0)$ indicates that we are looking at terms in the instanton expansion of order $Li_3(q_1^{k+l} q_2^k q_3^0)$.

In [14] (2.16) was explicitly checked for $k = 1$. We have calculated the N_{II} , now allowing both k and l to vary, to 20^{th} order for each threefold and recovered the coefficients of $-2E_4(q)E_6(q)/\eta(q)^{24}$, which exhibit M_{24} moonshine, as expected from (2.16). This constitutes a new numerical check of the duality at higher instanton number in the K3 fibers. We see explicitly that the connection to the M_{24} persists when both the K3's elliptic fiber and \mathbb{P}^1 base are “counted” multiple times.

The presence of $-2E_4(q)E_6(q)/\eta(q)^{24}$ in the STU model and its corresponding influence on the IIA side have been known for a long time. The first mirror symmetry computations of this type were done in [24], where the first few such Gromov-Witten invariants for X_2 were computed. As we see from our computations, these coefficients are also visible in the other X_n , indicating that the new connection to M_{24} is indeed independent of the instanton embedding on the heterotic side. We emphasize that the $S \rightarrow 0$ limit corresponds to a large base \mathbb{P}^1 on the IIA side, so the Gromov-Witten invariants relevant for moonshine come from worldsheet instantons mapping into the K3 fiber. This seemingly different connection between K3 and M_{24} certainly deserves further study and we point out in the conclusion that it could potentially extend to other K3 fibered CY_3 manifolds.

Having established the relationship between the sporadic group M_{24} and the Gromov-Witten invariants of the CY_3 manifolds X_n , as well as the holomorphic 3-form Ω of the mirror manifolds Y_n , we show in the next section that for (at least some of) the X_n part of the complex structure moduli space is also linked to M_{24} , and likewise for part of the Kähler moduli space of (at least some of) the Y_n . We also discuss which physical implications can be derived from such a connection. Here we mostly focus on the holomorphic 3-form Ω of the Y_n (and some of the X_n) and show in the section 4 that its relation to M_{24} leads to the appearance of dimension of M_{24} in the Gukov-Vafa-Witten [25] flux superpotential. In section 5, we show that for compactifications of the heterotic string theory on the X_n or Y_n the Yukawa couplings of the four dimensional $\mathcal{N} = 1$ theories are related to M_{24} .

3 Connecting both complex structure *and* Kähler moduli spaces to M_{24}

For $n = 2, 4, 6, 8, 12$ we can write the X_n as hypersurfaces in the weighted projective space $\mathbb{WP}_{1,1,n,2n+4,3n+6}$. For at least $n = 2, 4, 6, 12$ the mirror manifolds can be obtained from a Greene-Plesser construction, because the sum of the weights is divisible by each weight (see [19]). This means that we can quotient the space X_n by the maximal group of scaling symmetries to get a singular limit of its mirror, the Y_n manifold.

For example, for X_2 , the elliptic fibration over \mathbb{F}_2 , we have the Hodge numbers $h^{1,1} = 3$ and $h^{2,1} = 243$, where the three Kähler moduli correspond to the three STU moduli of the previous section. If we quotient by the maximal scaling symmetry $\mathbb{Z}_{12} \times \mathbb{Z}_{24}$ we project out 240 of the 243 complex structure moduli and leave the other three untouched. Resolving the orbifold singularities leads to 240 new Kähler moduli and the smooth Y_2 manifold with Hodge numbers $h^{1,1} = 243$ and $h^{2,1} = 3$. The interesting feature of this explicit construction is that one can clearly see that the 3 complex structure moduli of Y_2 have a moduli space that

is a subset of the 243 dimensional complex structure moduli space of X_2 . This subspace of the complex structure moduli space of X_2 is spanned by the three moduli that are invariant under the maximal group of scaling symmetries, the Greene-Plesser (GP) orbifold group. Let us identify them in the defining polynomial of X_2 . We can write X_2 as a hypersurface in $\mathbb{WP}_{1,1,2,8,12}$ (see for example the review [21]):

$$p = \frac{1}{24}(z_1^{24} + z_2^{24} + 2z_3^{12} + 8z_4^3 + 12z_5^2) - \psi_0 z_1 z_2 z_3 z_4 z_5 - \frac{1}{6}\psi_1 (z_1 z_2 z_3)^6 - \frac{1}{12}\psi_2 (z_1 z_2)^{12}, \quad (3.1)$$

where $z_i \in \mathbb{WP}_{1,1,2,8,12}$ and the three ψ_i are three of the 243 complex structure moduli.⁷ The other complex structure moduli correspond to deformations of the polynomial p that we have set to zero. As mentioned above, X_2 and therefore p can be quotiented by $G := \mathbb{Z}_{12} \times \mathbb{Z}_{24}$ leading to a singular limit of Y_2 . From the explicit action of the elements $(g_1, g_2) \in \mathbb{Z}_{12} \times \mathbb{Z}_{24}$:

$$\begin{aligned} g_1 : (z_1, z_2, z_3, z_4, z_5) &\rightarrow (e^{\frac{2\pi i}{12}} z_1, z_2, e^{\frac{2\pi i 11}{12}} z_3, z_4, z_5), \\ g_2 : (z_1, z_2, z_3, z_4, z_5) &\rightarrow (e^{\frac{2\pi i}{24}} z_1, e^{\frac{2\pi i 23}{24}} z_2, z_3, z_4, z_5), \end{aligned} \quad (3.2)$$

we see that p is invariant, and therefore the ψ_i correspond to the three complex structure moduli of the mirror manifold Y_2 that has Hodge numbers $h^{1,1} = 243$, $h^{2,1} = 3$. As we have shown in the previous section, these three complex structure moduli are connected to M_{24} and therefore the subset of the complex structure moduli space of X_2 that is spanned by the ψ_i is likewise connected to M_{24} . Thus we have implicated the Kähler moduli space *and* part of the complex structure moduli space of X_2 in Mathieu moonshine.

To recap, since we have connected the holomorphic 3-form Ω of all the Y_n with M_{24} , we can now conclude that for the X_n with at least $n = 2, 4, 6, 12$ there is also a connection between M_{24} and a subspace of the complex structure moduli space. Similarly, by mirror symmetry this then implies that for the Y_n with at least $n = 2, 4, 6, 12$ there is likewise a connection between M_{24} and a subspace of the Kähler moduli space.

Note that although the full hypermultiplet moduli spaces of X_n and Y_n are quaternionic Kähler, the special slices we discussed in this section (namely, the slice of the complex structure moduli space of X_n and the mirror slice of the Kähler moduli space of Y_n , and with all RR fields turned off) obey the relations of special Kähler geometry. This means for example, that we can calculate period vectors from a prepotential for X_2 (and likewise for X_n with $n = 4, 6, 12$). The other polynomial deformations that we have turned off in (3.1) will only appear in the computation of the eight G -invariant periods at higher orders, and can be consistently set to zero. This idea was first explored in [26] in the context of flux compactifications.

4 Mathieu representations in flux compactifications

Flux compactifications have been intensively studied during the last fifteen years due their great importance in solving the moduli problem in string compactifications [27, 28]. The

⁷In Appendix A, the ψ_i and numerical coefficients together are called a_i , with one a_i multiplying each monomial.

holomorphic 3-form Ω plays a central role in all flux compactifications on CY_3 manifolds that give rise to a four-dimensional $\mathcal{N} = 1$ theory due to the Gukov-Vafa-Witten superpotential [25]

$$W_{GVW} = \int_{CY_3} H_3 \wedge \Omega, \quad (4.1)$$

where H_3 denotes the NSNS 3-form flux. In flux compactifications of the heterotic string theory on any of the Y_n (or X_n for $n = 2, 4, 6, 12$) we therefore expect the appearance of M_{24} coefficients in the superpotential via the holomorphic 3-form Ω . (As we show in the next section, the superpotential arising in heterotic compactifications on the X_n and Y_n is also connected to M_{24} for $H_3 = 0$.)

For type II compactifications on a CY_3 manifold one has to do an orientifold projection in order to get a four-dimensional theory with $\mathcal{N} = 1$ supersymmetry. For example in type IIA one usually does an orientifold projection that gives rise to $O6$ -planes while in type IIB one chooses between either an $O3/O7$ or an $O5/O9$ orientifold projection.⁸ While these orientifold projections can project out some of the complex structure moduli contained in Ω , one generically expects that a connection to M_{24} survives. We work out the details for the most studied class of flux compactifications which is type IIB string theory on a CY_3 manifold in the presence of $O3/O7$ -planes. In that case the orientifold projection can potentially remove some entries of the period vector but usually all (or the majority) of the entries remain unaffected.

We follow the seminal paper [31] that constructs Minkowski vacua in which the complex structure moduli as well as the axio-dilaton are stabilized by fluxes. The reason is that one might wonder whether the appearance of dimensions of M_{24} in the holomorphic 3-form Ω are due to an actual symmetry of the Y_n and, if that were the case, whether such a symmetry could be a manifest symmetry of the vacua we find in flux compactifications. Due to the large order of M_{24} which is $|M_{24}| \approx 2 \times 10^9$ such a symmetry would be very surprising and tremendously interesting. That a sporadic group appears as symmetry group of the internal space used in a string compactifications is of course at the heart of Monstrous moonshine [32]. Monstrous moonshine is essentially explained by the fact that the \mathbb{Z}_2 orbifold of \mathbb{R}^{24}/Λ , where Λ is the Leech lattice, has as its symmetry group the Monster group. Compactifying the (left-moving) bosonic string theory on this space leads to a theory with Monster symmetry and the partition function, which is Klein's J -function, can therefore be expanded in such a way that the coefficients are (sums of) irreducible representations of the Monster group. Likewise it is clear that the newly discovered mock modular moonshine phenomena involving the Mathieu groups M_{22} and M_{23} [33] tell us that superstring compactifications on asymmetric \mathbb{Z}_2 orbifolds of $\mathbb{R}^8/\Lambda_{E_8}$, with Λ_{E_8} denoting the E_8 root lattice, have the symmetry group M_{22} or M_{23} . For the case of Mathieu moonshine, however, things are not yet understood and there does not seem to be a direct connection between the Mathieu group M_{24} and the symmetry groups

⁸Depending on the orientifold projection, the four-dimensional $\mathcal{N} = 1$ theory might also contain vector multiplets. For type IIB compactifications the resulting holomorphic gauge kinetic function is also related to the holomorphic 3-form Ω and therefore to M_{24} [29, 30].

of non-linear sigma models with $\mathcal{N} = (4, 4)$ worldsheet symmetry and $K3$ target space [11]. Thus, the fascinating connection between the Gromov-Witten invariants of the X_n and the periods of the holomorphic 3-form Ω of the Y_n is currently not understood. Nevertheless, it is interesting to understand whether such a symmetry, if it is found to exist, would remain unbroken in flux compactifications. This is what we are explicitly doing for the case of type IIB flux compactifications.

In type IIB flux compactifications on CY_3 manifolds we can turn on the NSNS 3-form flux H_3 and the RR 3-form flux F_3 . It is useful to combine these into the complex flux $G_3 = F_3 - \tau H_3$, where $\tau = C_0 + ie^{-\phi}$ is the axio-dilaton. We can expand the G_3 flux in the basis (A.6) as

$$G_3 = (M^i - \tau \tilde{M}^i) \alpha_i - (N_j - \tau \tilde{N}_j) \beta^j, \quad i = 0, 1, \dots, h^{2,1}. \quad (4.2)$$

Introducing the flux vectors $f = (N_i, -M^I, -M^0)$ and $h = (\tilde{N}_i, -\tilde{M}^I, -\tilde{M}^0)$ where $I = 1, 2, \dots, h^{2,1}$, we can write the full flux superpotential as

$$W = \int_{CY_3} G_3 \wedge \Omega = (f - \tau h) \cdot \Pi, \quad (4.3)$$

where the period vector Π is given in (2.15). As we have argued by duality, the instanton numbers (cf. (2.16)) that appear at different powers of q_i in the period vector (2.15) are related to sums of dimensions of *different* irreducible representations of M_{24} . Therefore it seems clear that Π does not transform in any well defined way under a potential M_{24} symmetry group. We also notice from equation (4.3) that Π is contracted with a fixed flux vector. This flux vector arises from the expansion of the fluxes in term of 3-forms (4.2) and may consist of arbitrary integers, provided they satisfy the tadpole cancellation condition. Since there does not seem to be any M_{24} symmetry acting on the third cohomology class of the X_n or Y_n (cf. (A.6)), the flux vector should be invariant under any potential M_{24} symmetry.

So the lack of a well defined transformation of Π together with the contraction with the invariant flux vectors clearly breaks any potential M_{24} symmetry of the X_n or Y_n . Thus the resulting flux vacua do therefore not have in any obvious way a large sporadic symmetry group. However, this by no means excludes the exciting possibility that one could define an M_{24} action on the curves that give rise to the Gromov-Witten invariants that seem to be connected the M_{24} .

5 Mathieu representations in Yukawa couplings

Compactifications of the heterotic string theory on CY_3 manifolds give rise to four dimensional $\mathcal{N} = 1$ theories with a variety of gauge groups and chiral matter. These compactifications have been studied for decades and have been textbook material for a long time [34]. Here we review a few basic facts and show explicitly how the connection between M_{24} and the Gromov-Witten invariants as well as the holomorphic 3-form Ω manifests itself in the Yukawa couplings

of the four-dimensional theories obtained from compactifying the heterotic string theory on the X_n or Y_n .

For compactifications of the heterotic $E_8 \times E_8$ string theory on a CY_3 manifold M we have to solve the H_3 Bianchi identity which in the absence of NS5-branes reads

$$dH_3 = \frac{\alpha'}{4} [\text{Tr}(R_2 \wedge R_2) - \text{Tr}_V(F_2 \wedge F_2)] . \quad (5.1)$$

If we set the gauge connection equal to the spin connection, then this equation is trivially satisfied and all other equations of motion are equally satisfied for $H_3 = 0$ and constant string coupling. The resulting four dimensional theory preserves $\mathcal{N} = 1$ supersymmetry and has a vanishing cosmological constant. Equating the spin and the gauge connection breaks one of the E_8 's to an E_6 GUT group and leaves a second unbroken E_8 . These gauge groups can be further broken by modding out by discrete groups and turning on Wilson lines or by giving expectation values to certain moduli. However, we refrain from doing so to keep the presentation of the connection to M_{24} as transparent as possible. It would be interesting to check whether more involved compactifications on the X_n or Y_n can give rise to semi-realistic models while still preserving the connection to M_{24} .

The low energy effective action and the number of chiral multiplets in these compactifications are determined by the topological data of the CY_3 manifold M . Denoting the Hodge numbers by $h^{p,q}$ one finds $h^{1,1}$ chiral multiplets Ψ^i in the **27** of E_6 and $h^{2,1}$ chiral multiplets Φ^α in the $\overline{\mathbf{27}}$ of E_6 [34].⁹ In addition there are several uncharged chiral multiplets like the $h^{1,1}$ Kähler moduli t^i , the $h^{2,1}$ complex structure moduli u^α and the axio-dilaton s whose vacuum expectation value controls the tree-level holomorphic gauge kinetic coupling $f^{tree} = s$. The Kähler potential for the uncharged Kähler and complex structure moduli as well as the axio-dilaton is given by¹⁰

$$K_1(t, \bar{t}) = -\ln \left(\frac{1}{6} \int_M J \wedge J \wedge J \right) = -\ln \left(-\frac{i}{6} \kappa_{ijk}^0 (t^i - \bar{t}^i)(t^j - \bar{t}^j)(t^k - \bar{t}^k) \right) , \quad (5.2)$$

$$K_2(u, \bar{u}) = \ln \left(i \int_M \Omega(u) \wedge \bar{\Omega}(\bar{u}) \right) , \quad (5.3)$$

$$K_3(s, \bar{s}) = -\ln(s + \bar{s}) . \quad (5.4)$$

The Kähler potential for the matter fields Ψ^i and Φ^α is

$$K_{matter} = e^{\frac{\kappa_2 - \kappa_1}{3}} \frac{\partial^2 K_1(t, \bar{t})}{\partial t^i \partial \bar{t}^j} \Psi^i \bar{\Psi}^j + e^{\frac{\kappa_1 - \kappa_2}{3}} \frac{\partial^2 K_2(u, \bar{u})}{\partial u^\alpha \partial \bar{u}^\beta} \Phi^\alpha \bar{\Phi}^\beta . \quad (5.5)$$

We see that the holomorphic 3-form Ω appears in the Kähler potential of the four-dimensional theory and therefore M_{24} irreps will appear in the kinetic terms for the u^α and Φ^α in compactifications on the Y_n . Even more interesting is the superpotential. There are non-zero

⁹Here we use different conventions than [34] for ease of presentation.

¹⁰We slightly abuse the notation and label the multiplets and the scalar field in the multiplet by the same letter.

Yukawa couplings for the matter fields that depend on the vacuum expectation values of the uncharged moduli. In particular the superpotential takes the form

$$\begin{aligned} W(t, u, \Psi, \Phi) &= \frac{1}{6} \kappa_{ijk}^0[M] \Psi^i \Psi^j \Psi^k + \frac{1}{6} \frac{\partial^3 F(u)}{\partial u^\alpha \partial u^\beta \partial u^\gamma} \Phi^\alpha \Phi^\beta \Phi^\gamma \\ &= \frac{1}{6} \kappa_{ijk}^0[M] \Psi^i \Psi^j \Psi^k + \frac{1}{6} \bar{\kappa}_{\alpha\beta\gamma}[M] \Phi^\alpha \Phi^\beta \Phi^\gamma, \end{aligned} \quad (5.6)$$

where the gauge indices are contracted with the E_6 invariants. We see that the Yukawa couplings for the Φ^α are derivatives of the prepotential. For compactifications with $M = Y_n$ these are therefore directly related to M_{24} . The above Kähler and superpotential receive non-perturbative instanton corrections. In particular one expects that the Kähler potential $K_1(t, \bar{t})$ and the superpotential for the Ψ^i receive corrections. Due to the invariance under mirror symmetry of these compactifications that preserve $(2, 2)$ worldsheet supersymmetry, we expect that these corrections are exactly such that $\kappa_{ijk}^0[M]$ becomes $\kappa_{ijk}[M]$ (cf. equation (A.11)). This means that for compactifications on the X_n the Yukawa couplings for the fields transforming as **27** are connected to the Mathieu group M_{24} as well, due to the connection between the Gromov-Witten invariants that appear in the instanton-corrected triple intersection numbers and M_{24} . As we have argued before at least for $n = 2, 4, 6, 12$ there is also a connection between the holomorphic 3-form of the X_n and the Gromov-Witten invariants of the Y_n , so at least for these spaces we expect M_{24} to play a role in both Yukawa couplings.

For compactifications of the heterotic string theory on the Y_n , we can explicitly calculate the Yukawa couplings in the STU basis up to non-perturbative corrections in S , which makes the connection to M_{24} quite transparent. We find the following Yukawa couplings $\bar{\kappa}_{\alpha\beta\gamma}[Y_n] = \partial_\alpha \partial_\beta \partial_\gamma F(S, T, U)$ with $F(S, T, U)$ given in (2.8) (cf. also [35, 36])

$$\begin{aligned} \bar{\kappa}_{STU}[Y_n] &= 1, \\ \bar{\kappa}_{UUU}[Y_n] &= 2 - 2 \sum_{\substack{k>0, l \in \mathbb{Z} \\ k=0, l>0}} c(kl) l^3 \left(\frac{1}{1 - q_T^k q_U^l} - 1 \right) = -2 \frac{E_4(q_U) E_4(q_T) E_6(q_T)}{\eta(q_T)^{24} (J(q_U) - J(q_T))}, \\ \bar{\kappa}_{TTT}[Y_n] &= -2 \sum_{k>0, l \in \mathbb{Z}} c(kl) k^3 \left(\frac{1}{1 - q_T^k q_U^l} - 1 \right) = -2 \frac{E_4(q_T) E_4(q_U) E_6(q_U)}{\eta(q_U)^{24} (J(q_T) - J(q_U))}, \\ \bar{\kappa}_{UUT}[Y_n] &= -2 \sum_{k>0, l \in \mathbb{Z}} c(kl) l k^2 \left(\frac{1}{1 - q_T^k q_U^l} - 1 \right), \\ \bar{\kappa}_{UTT}[Y_n] &= -2 \sum_{k>0, l \in \mathbb{Z}} c(kl) l^2 k \left(\frac{1}{1 - q_T^k q_U^l} - 1 \right), \end{aligned} \quad (5.7)$$

where we used the fact that $\partial_x^3 Li_3(e^x) = \frac{e^x}{1-e^x}$. All other Yukawa couplings vanish perturbatively in S . For $\bar{\kappa}_{TTT}[Y_n]$ and $\bar{\kappa}_{UUU}[Y_n]$ a closed form was given in [35]. There it was also argued that $\bar{\kappa}_{UUT}[Y_n]$ and $\bar{\kappa}_{UTT}[Y_n]$ likewise have a pole for $T = U$ that goes like $(J(q_U) - J(q_T))^{-1}$. However, we did not try to find a closed form for the latter two since the sums make the connection to M_{24} much more transparent. (Recall that the connection between M_{24} and the Yukawa couplings arises due to the relation between the $c(m)$ defined

in (2.9) and M_{24} ; see section 2). We thus see that perturbatively in S all non-zero Yukawa couplings, except the trivial $\bar{\kappa}_{STU}[Y_n]$, are linked to M_{24} .

From the explicit calculation of the periods that we do in the appendix, we can get the Yukawa couplings to arbitrarily high powers in q_U , q_T as well as q_S and we spell them out to a certain order in appendix B. It is a natural question to ask whether these non-perturbative corrections in S are likewise related to the Mathieu group M_{24} . As explained in [14], based on the recursion relation derived in [22, 23], one expects that the answer is yes. Explicitly, on the type IIA side these corrections to the prepotential F that are non-perturbative in S are determined by equations that use as seed the term in F that is perturbative in S and linear in $e^{-2\pi(T-U)}$. This term is nothing but $-2E_4E_6/\eta^{24}$ which is directly related to M_{24} as explained in section 2. Thus we see that essentially all terms in the Yukawa couplings are implicated in Mathieu moonshine (albeit in a potentially complicated way).

6 Conclusion

Mathieu moonshine is an intriguing and not yet understood connection between the elliptic genus of $K3$ and the largest Mathieu group M_{24} . In this short paper we extend previous results and explicitly exhibit a link between the periods of certain CY_3 manifolds and M_{24} . In particular, based on string dualities it was argued in [14] that the Gromov-Witten invariants of the CY_3 manifolds X_n , that are elliptic fibrations over \mathbb{F}_n , exhibit a connection to M_{24} . We extended the checks of this duality that were performed in [14] and argued that this then implies a link between the holomorphic 3-form Ω of the mirror manifolds Y_n and M_{24} . Based on the explicit construction of mirror pairs we have shown that (at least for $n = 2, 4, 6, 12$) there is a subspace of the complex structure moduli space for the X_n that is likewise related to M_{24} . This then directly implies that a subset of the Gromov-Witten invariants of the Y_n (for at least $n = 2, 4, 6, 12$) are also connected to M_{24} .

These connections lead to a variety of interesting implications, two of which we discussed in detail. Firstly, flux compactifications on the CY_3 manifolds that are implicated in Mathieu moonshine lead to superpotentials with coefficients that are related to the dimensions of representations of M_{24} . We noted however that even if these CY_3 manifolds have an underlying M_{24} symmetry, then this symmetry should be broken by the Gukov-Vafa-Witten superpotential. Secondly, for simple compactifications of the heterotic $E_8 \times E_8$ string theory on the CY_3 manifolds connected to M_{24} , we have shown that the Yukawa couplings of the matter fields have an interesting connection to M_{24} . In these theories this thus leads to a relation between particle masses and dimensions of representations of the largest Mathieu group M_{24} .

It would be interesting to find and study further such connection between physical quantities in four dimensional theories and the Mathieu group M_{24} . For example, the action of supersymmetric D6-branes wrapping 3-cycles inside a CY_3 manifold involves integrals over the holomorphic 3-form Ω [37]. This should lead to a relation between M_{24} and intersecting D6-brane models for compactifications on the CY_3 manifolds whose periods are connected to M_{24} .

Interestingly, we noticed that $E_4 E_6 / \eta^{24}$ also governs a subclass of Ooguri-Vafa invariants of the three-modulus system composed of the degree-18 CY_3 in $\mathbb{WP}_{1,1,1,6,9}$ and a particular A-brane. See Section 3.2 of [38] for details of this setup. We noticed that for certain world-sheets wrapping the elliptic fiber of the CY_3 , these open string analogues of Gromov-Witten invariants were given by exactly $E_4 E_6 / \eta^{24}$. We computed these invariants to tenth order as a simple check. On the B-model side, the computation of these invariants could be mapped to computations of the periods of a certain $K3$ given as a hypersurface in $\mathbb{WP}_{1,1,4,6}$, much like the $K3$ fiber of the CY_3 manifolds studied in this paper! Therefore, it is natural to ask if there is a geometrical explanation for the appearance of this modular form in the periods of these special $K3$ s. Of course, the symplectic automorphisms of such $K3$ s are strictly subgroups of M_{23} , so such an explanation is far from obvious. We may at least be able to understand its appearance using restrictions from modularity. While we think such a question is of interest in understanding M_{24} 's connection to $K3$ surfaces, it may have further implications for string compactifications as well. In particular, it may suggest that more CY_3 s (possibly with brane) containing such a $K3$ fiber (or submanifold, up to a change in variables) will have some of its enumerative geometry governed by moonshine. As we discussed in this paper, these invariants manifest in certain quantities in type II and heterotic compactifications.

Relatedly, in [39] the authors observe that the dimensions of irreducible representations of M_{24} seem to appear in the stable pair invariants of $K3$ fibered CY_3 manifolds. This seems to provide another link between the geometry of $K3$ -fibered CY_3 manifolds and Mathieu moonshine and it would be very interesting to explore potential connections to our work via the Gromov-Witten/stable pairs correspondence. For example, we do not yet understand how to “twine” our Gromov-Witten invariants by simple geometric symmetries, and so we cannot compute twining genera to support the connection between moonshine and geometry. The work of [39], however, may suggest natural geometric twinings, perhaps analogous to the eta-product twinings computed in Mason’s moonshine, which would realize an interesting subgroup of M_{24} symmetries acting directly on geometric invariants. This would also be fascinating from the spacetime perspective, as it would translate to an M_{24} action on the algebra of BPS states.

Recently two new moonshine phenomena were discovered in [33]. It would be very interesting to understand how they can be connected to explicit string theory compactifications. This should undoubtedly give rise to new interesting physical and mathematical connections involving the Mathieu groups M_{22} and M_{23} .

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A Brief Review of Mirror Symmetry

In this section, we will provide a brief review of some of the basic techniques in mirror symmetry that we used in our computations. Our presentation will mostly follow [24, 40, 41] and will use the notation of [41]. For a comprehensive review of mirror symmetry, see the excellent text [42]. For an explicit computation in the one-modulus example of the quintic, we refer the reader to the seminal paper [43].

Mirror symmetry relates the A- and B-model topological string theories on the mirror manifolds X_n, Y_n . The A-model is sensitive only to Kähler deformations and hence computes the Gromov-Witten invariants on X_n ; the B-model probes the complex structure moduli space through variations of the Hodge structure. The mirror manifolds are topologically distinct, and their Hodge structures map to one another via a diagonal reflection on their Hodge diamonds. One computes a “mirror map” t^i , which is nothing but a special set of local coordinates, to relate the two theories.

A.1 Toric data

In this paper, we focus on closed string mirror symmetry between two CY_3 manifolds representable as hypersurfaces in toric varieties. The hypersurfaces are specified by reflexive rational convex polyhedra (Δ, Δ^*) and their associated rational fans. The polyhedra will contain the origin, which we denote ν_0 . Other integral points in Δ , including vertices, will be denoted ν_i . Given a *reflexive* polyhedron Δ as a function of the weights, w_i , of an ambient weighted projective space $W = \mathbb{WP}_{w_1, w_2, w_3, w_4, w_5}$ one can construct its dual, Δ^* , which specifies the topological data of the mirror Calabi-Yau. This is a convenient algorithmic language for finding mirror manifolds which reproduces and extends the Greene-Plesser procedure, which constructs mirrors by orbifolding X_n by some abelian group [19] (see also [44] for a procedure to find mirrors away from the Fermat point). For example, Batyrev [45] found a simple formula computing the Hodge numbers of the mirror pair in terms of the numbers of integral points on the faces and interiors of the polyhedra.

If the polyhedron is Gorenstein¹¹, as are the $X_n, n = 2, 4, 6, 12$, the dual is simply given by:

$$\Delta^*(w) := \left\{ (x_1, \dots, x_5) \in \mathbb{R}^5 \mid \sum_{i=1}^5 w_i x_i = 0, x_i \geq -1 \right\}. \quad (\text{A.1})$$

In this case, the origin is the only interior point of Δ . Note that the polyhedra satisfy $(\Delta^*)^* = \Delta$.

Normally, we define a hypersurface in weighted projective space as the zero locus of a quasi-homogeneous polynomial $p(z) = 0$, which will be nonsingular if it satisfies the transversality conditions. That is, it never fulfills $p(z_p) = dp(z_p) = 0$ for any point z_p . We can define a toric hypersurface in W^* as the zero locus of the Laurent polynomial

¹¹The polyhedron will be Gorenstein if the least common multiple of all the weights w_i divides the degree d of the hypersurface.

$f_{\Delta^*}(a, X) = a_0 - \sum_i a_i X^{\nu_i^*}$, $f_{\Delta^*} \in \mathbb{C}(X_1^{\pm 1}, \dots, X_4^{\pm 1})$, where ν_i^* are the integral points in Δ^* and a_i are complex constants parametrizing the complex structure deformations of the B-model geometry. We have used the notation $X^{\nu_i^*} := \prod_{j=1}^4 X_j^{\nu_{i,j}^*}$. Similar definitions hold for the dual (unstarred) quantities.

For some Calabi-Yau X , its complex structure moduli space is encapsulated by lattice points in the polyhedron Δ^* . Each lattice point corresponds to a monomial perturbation. Points in the dual polyhedron correspond to exceptional divisors and therefore encode the Kähler moduli space. Mirror symmetry says that if two Calabi-Yaus X and Y are a mirror pair, each realized by a toric hypersurface as described above, then the polyhedra associated to X , (Δ_X, Δ_X^*) , are isomorphic to the polyhedra associated to Y , (Δ_Y^*, Δ_Y) . This exchanges the complex structure and Kähler moduli spaces. For simplicity of notation, we have dropped the X, Y subscripts above and in what follows, since we will only care about the pair (Δ_X, Δ_X^*) . In this way, we differ slightly from the notation of [41], but hope our meaning is clear.

The last important toric quantity to introduce is the Mori cone. There are $5 + h^{2,1}$ integral points ν_i^* , including the origin ν_0^* that do not lie in the interior of faces of codimension one. These are the points that we used to construct the Laurent polynomial above. We define a lattice of relations of the form $\sum_i l_i \nu_i^* = 0$, $l_i \in \mathbb{Z}$. There are $h^{2,1}(Y_n)$ ($= 3$ for our computations on the B-model side) generators of this lattice. Once we find this lattice, we define extended vectors $(l_0^\alpha, \{l_i^\alpha\}) := (-\sum_i l_i^\alpha, \{l_i^\alpha\})$. The Mori cone generates the lattice of relations and it will show up in the computation of the periods.

A.2 Periods and Picard-Fuchs equations

With topological data in hand, we may now study the B-model on Y_n to extract its holomorphic $(3, 0)$ form Ω and compute the periods thereof. Following the previous section, this is the mirror manifold associated to Δ^* so we will explicitly use the $*$ notation to label toric quantities. The period integrals are given by

$$\Pi(a) = \int_{\gamma_i} \frac{a_0}{f(a, X)} \prod_{j=1}^n \frac{dX_j}{X_j}, \quad (\text{A.2})$$

where $f_{\Delta^*}(a, X) = a_0 - \sum_i a_i X^{\nu_i^*}$ is the defining polynomial for the hypersurface in terms of complex structure moduli a_i and X_j are inhomogeneous coordinates on $(\mathbb{C}^*)^4$ in the ambient projective space. We have again employed the common notation $X^{\nu_i^*} := \prod_j X_j^{\nu_{i,j}^*}$. The number of periods is $\dim(H^3) = 2(h^{2,1}(Y_n) + 1) = 2(h^{1,1}(X_n) + 1)$, which equals 8 for X_n being an elliptic fibrations over \mathbb{F}_n and $n = 0, 1, 2$.

The periods are solutions to the Picard-Fuchs equations and are readily computable in the large complex structure limit, or around the point of maximal unipotent monodromy. This point will be mapped to the large radius limit of X_n via the mirror map. Given the Mori cone and complex structure moduli, it is convenient to define the variables $u_\alpha := \prod a_i^{l_i^\alpha}$, $\alpha = 1, \dots, h^{1,1}(X_n)$. The large complex structure point is then $u_\alpha = 0$. First, one computes the fundamental period directly by choosing the cycle $\Gamma = \{(X_1, X_2, X_3, X_4 \in \mathbb{C}^4) \mid |X_i| = 1\}$ and

computing the integral in the $a_0 \rightarrow \infty$ limit. The result is

$$w_0(u) = \sum_{n_\alpha} \frac{(-\sum_\alpha l_0^\alpha n_\alpha)!}{\prod_{i>0} (l_i^\alpha n_\alpha)!} \prod_\alpha u_\alpha^{n_\alpha}, \quad (\text{A.3})$$

where the sum is such that the integral n_α do not let the arguments of the factorials become non-negative.

Now we may set up the GKZ hypergeometric system of partial differential equations which the fundamental period satisfies and a subset of this solution space is the solution space of the Picard-Fuchs (PF) system itself. By examining recursion relations satisfied by the coefficients of the fundamental period, one can find linear differential operators that annihilate the periods:

$$\left(p_\beta(u_\alpha \frac{d}{du_\alpha}, u_\beta \frac{d}{du_\beta}) - u_\beta q_\beta(u_\alpha \frac{d}{du_\alpha}, u_\beta \frac{d}{du_\beta}) \right) w(u) = 0, \quad (\text{A.4})$$

where p and q are polynomials in the logarithmic derivatives shown. One may then extract the PF system from this GKZ system (sometimes with difficulty, though it is straightforward in our case).

Now, a variation of Hodge structure will change the type of $\Omega(u)$. We can write the cohomology class $H^3(Y_n) = \bigoplus_{p=0}^3 H^{3-p,p}$ by Hodge decomposition, which will vary over the moduli space of complex structures. Indeed, one may think of $H^3(Y_n)$ as the fiber of a vector bundle over the moduli space of complex structures, equipped with a flat connection called the Gauss-Manin connection. One can derive this connection from the PF equations but we will not do so here. For our purposes, we note that we can identify derivatives of Ω with Hodge filtration spaces and can find linear combinations of derivatives that span the whole filtration. The dimensions of the spaces $(F^3, F^2/F^3, F^1/F^2, F^0/F^1)$ are $(1, h^{2,1}, h^{2,1}, 1)$ and integrating the vector obtained from a section of this filtration gives the period vector. Note that the entry corresponding to the 1-dimensional filtration space F^3 is, of course, $\Omega(u)$ itself, and the other entries are logarithmic derivatives thereof.

Let's find the vector of periods from the PF equations more concretely, around the point $u = 0$. If we apply the method of Frobenius to the PF equations around this point, the result is one power series solution (the fundamental period), and logarithmic solutions, up to a gauge transformation. We analytically continue the fundamental period by swapping the factorials for gamma functions and we add $h^{2,1}$ new variables ρ_α such that $w_0(u, \rho) = \sum c(n + \rho) u^{n+\rho}$. We recover the fundamental period by setting $\rho = 0$. In the language of Frobenius, ρ are called the indices, or solutions to the indicial equations, and they turn out to be maximally degenerate and zero at the point of maximal unipotent monodromy. Turning the crank, we find that the period vector is

$$\Pi = \begin{pmatrix} w_0(u) \\ \frac{1}{2\pi i} \partial_{\rho_i} w_0|_{\rho=0} \\ \frac{1}{2} \frac{1}{(2\pi i)^2} \sum \kappa_{ijk}^0 [X_n] \partial_{\rho_j} \partial_{\rho_k} w_0|_{\rho=0} \\ \frac{-1}{6} \frac{1}{(2\pi i)^3} \sum \kappa_{ijk}^0 [X_n] \partial_{\rho_i} \partial_{\rho_j} \partial_{\rho_k} w_0|_{\rho=0} \end{pmatrix}. \quad (\text{A.5})$$

Note that the dimensions are $(1, h^{2,1}, h^{2,1}, 1)$ as promised. The constants $\kappa_{ijk}^0[X_n]$ turn out to be the classical triple intersection numbers of X_n in a particular basis.

A.3 Flat coordinates and the mirror map

Before we discuss the mirror map, we first introduce the symplectic basis of $H^3(Y_n, \mathbb{Z})$. Since the moduli space of complex structures enjoys the properties of special geometry, this will be the appropriate basis to reexpress the periods in terms of the holomorphic prepotential. As usual, it is

$$\int_{A^i} \alpha_i = - \int_{B_i} \beta^j = \int_{Y_n} \alpha_i \wedge \beta^j = \delta_i^j, \quad i = 0, 1, \dots, h^{2,1}. \quad (\text{A.6})$$

In this basis, the periods are written as

$$z^i = \int_{A^i} \Omega, \quad \mathcal{F}_i(z) = \int_{B_i} \Omega. \quad (\text{A.7})$$

The z^i are the special projective coordinates on the moduli space (not to be confused with the coordinates of W) and will be identified with $w_i(u)$. Griffiths transversality gives the condition $\int \Omega \wedge \frac{\partial \Omega}{\partial z^i} = 0$, which implies $\mathcal{F}_i = \frac{\partial \mathcal{F}}{\partial z^i}$, where \mathcal{F} is the holomorphic prepotential. We can go to a physical gauge by dividing by z^0 and defining new coordinates $t^i = z^i/z^0$.

In this basis, the triple intersection numbers are $\bar{\kappa}_{ijk} = \int \Omega \wedge \frac{\partial^3}{\partial t^i \partial t^j \partial t^k} \Omega$. Moreover, the period vector becomes

$$\begin{pmatrix} 1 \\ t^i \\ \frac{\partial}{\partial t^i} \frac{\mathcal{F}}{(z^0)^2} \\ 2 \frac{\mathcal{F}}{(z^0)^2} - t^i \frac{\partial}{\partial t^i} \frac{\mathcal{F}}{(z^0)^2} \end{pmatrix}. \quad (\text{A.8})$$

The mirror map is given by identifying the new coordinates t^i with the solutions of the PF equations that are linear in logarithms (i.e. the first subspace of dimension $h^{2,1}$):

$$t^i(u) = \frac{w_i(u)}{w_0(u)}. \quad (\text{A.9})$$

A.4 Triple intersection numbers and Gromov-Witten invariants

As discussed in the previous section, the triple intersection numbers $\bar{\kappa}_{ijk}$ are readily computed once we have found the periods. In terms of the prepotential, these are simply rewritten as $\sum_{l=0}^{h^{2,1}} (z^l \partial_i \partial_j \partial_k \mathcal{F}_l - \mathcal{F}_l \partial_i \partial_j \partial_k z^l)$. We now wish to find the triple intersection numbers on the mirror manifold X_n .

If we define $\mathcal{F} = w_0^2 F$, they are:

$$\kappa_{ijk}[X_n] = \partial_{t^i} \partial_{t^j} \partial_{t^k} F(t) = \frac{1}{w_0(u(t))^2} \frac{\partial u_\alpha}{\partial t^i} \frac{\partial u_\beta}{\partial t^j} \frac{\partial u_\gamma}{\partial t^k} \bar{\kappa}_{\alpha\beta\gamma}[Y_n](u(t)). \quad (\text{A.10})$$

If we wish to express the triple intersection numbers in terms of t^i , which we know to be the Kähler moduli in the limit of large radius, we must invert the mirror map. To do this, we define the variable $q_j = e^{2\pi i t^j}$. Then we can perform a series inversion $u_i(t)$ fairly laboriously order-by-order. For the simple example of the quintic, this is outlined nicely in [46]. For our three-modulus Hirzebruch surfaces, this is best done with a computer program like Mathematica [47].

We can write these full instanton corrected triple intersection numbers as

$$\kappa_{ijk}[X_n] = \kappa_{ijk}^0[X_n] + \sum_{n_i} \frac{N(\{n_i\}) n_i n_j n_k \prod_l q_l^{n_l}}{1 - \prod_l q_l^{n_l}}, \quad (\text{A.11})$$

where $n_i = \int_C h_i \in \mathbb{Z}$, $h_i \in H^{1,1}(X_n, \mathbb{Z})$. This expression comes from performing a geometric series coming from multiple coverings of the curve C . The integers $N(\{n_i\})$ then count the number of (isolated, non-singular) rational curves C of degree $\{n_i\}$. Hence, these are the integral genus-zero Gromov-Witten invariants. This expression follows from the geometrical definition of the corrected triple intersection numbers, using the fact that $\int_C J = \sum t^i n_i$, where J is the Kähler form.

We note that the classical contribution to the triple intersection numbers, κ_{ijk}^0 , are given in a basis corresponding to the variables u_α . It is easy to compute them in the basis of harmonic $(1,1)$ forms $h_J, h_{D_1}, \dots, h_{D_{h^{1,1}-1}}$, which correspond to the complex structure moduli a_i . In the toric language, the computation is described explicitly in [24]. To compute them in the basis of divisors (or harmonic forms) corresponding to the u variables, we perform the change of variables $h_J = h_1$, $h_{D_i} = \sum_\alpha l_{i+5}^\alpha h_\alpha$.

Lastly, we note that the prepotential can then be written as

$$\mathcal{F} = (z_0)^2 \left(\frac{\kappa_{ijk}^0[X_n]}{6} t^i t^j t^k + (1/2) a_{ij} t^i t^j + b_i t^i + c/2 + \frac{1}{(2\pi i)^3} \sum_{(n_i)} N_{(n_i)} Li_3(q^{(n_i)}) \right), \quad (\text{A.12})$$

where, up to monodromy transformations, $a_{ij} = 0$, $b_i = \frac{1}{24} \int_{X_n} c_2 \wedge h_i$, $c = \frac{1}{(2\pi i)^3} \chi(X_n) \zeta(3)$. Substituting this expression into the period vector makes the dependence of the periods on the Gromov-Witten invariants manifest.

B Data for Elliptically Fibered Threefolds

Here we present some results of our mirror symmetry computations for elliptic fibrations over \mathbb{F}_n , $n = 0, 1, 2$. We list the Mori cone generators, classical topological ring, and the Fourier expansion of the triple intersection numbers $\kappa_{ijk}[X_n] = \bar{\kappa}_{ijk}[Y_n]$ to 5^{th} order in the moduli q_1, q_2, q_3 that are related to the STU moduli as indicated below. From this expansion, one can easily read off the Gromov-Witten invariants, via (A.11). For all three manifolds, $\chi(X_n) = -480$, $n = 0, 1, 2$.

We also list our $b_i = \frac{1}{24} \int_{X_n} c_2 \wedge h_i$, expressed in the same basis of h_i as [41], which we describe in A. Observables like the flux superpotential are, of course, independent of basis choices.

B.1 \mathbb{F}_0

$$l^1 = \begin{pmatrix} -6 & 3 & 2 & 1 & 0 & 0 & 0 \end{pmatrix} \quad l^2 = \begin{pmatrix} 0 & 0 & 0 & -2 & 1 & 1 & 0 \end{pmatrix} \quad l^3 = \begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 1 \end{pmatrix} \quad (\text{B.1})$$

$$q_1 = q_U, \quad q_2 = \frac{q_S}{q_U}, \quad q_3 = \frac{q_T}{q_U}. \quad (\text{B.2})$$

$$24b_1 = 92, \quad 24b_2 = 24, \quad 24b_3 = 24. \quad (\text{B.3})$$

$$\kappa_{111}^0[X_n] = 8, \quad \kappa_{112}^0[X_n] = 2, \quad \kappa_{113}^0[X_n] = 2, \quad \kappa_{123}^0[X_n] = 1. \quad (\text{B.4})$$

$$\begin{aligned} \kappa_{111}[X_n] = & 8 + 480q_1 + 4320q_1^2 + 13440q_1^3 + 35040q_1^4 + 60480q_1^5 + 480q_1q_2 \\ & + 2263104q_1^2q_2 + 460581120q_1^3q_2 + 30561073920q_1^4q_2 + 4320q_1^2q_2^2 + 460581120q_1^3q_2^2 \\ & + 480q_1q_3 + 2263104q_1^2q_3 + 460581120q_1^3q_3 + 30561073920q_1^4q_3 + 1440q_1q_2q_3 \\ & - 1808640q_1^2q_2q_3 + 1390953600q_1^3q_2q_3 + 2400q_1q_2^2q_3 - 3617280q_1^2q_2^2q_3 + 3360q_1q_2^3q_3 \\ & + 4320q_1^2q_3^2 + 460581120q_1^3q_3^2 + 2400q_1q_2q_3^2 - 3617280q_1^2q_2q_3^2 + 16800q_1q_2^2q_3^2 \\ & + 3360q_1q_2q_3^3 + \dots \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} \kappa_{112}[X_n] = & 2 + 480q_1q_2 + 1131552q_1^2q_2 + 153527040q_1^3q_2 + 7640268480q_1^4q_2 + 4320q_1^2q_2^2 \\ & + 307054080q_1^3q_2^2 + 1440q_1q_2q_3 - 904320q_1^2q_2q_3 + 463651200q_1^3q_2q_3 + 4800q_1q_2^2q_3 \\ & - 3617280q_1^2q_2^2q_3 + 10080q_1q_2^3q_3 + 2400q_1q_2q_3^2 - 1808640q_1^2q_2q_3^2 + 33600q_1q_2^2q_3^2 \\ & + 3360q_1q_2q_3^3 + \dots \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} \kappa_{113}[X_n] = & 2 + 480q_1q_3 + 1131552q_1^2q_3 + 153527040q_1^3q_3 + 7640268480q_1^4q_3 \\ & + 1440q_1q_2q_3 - 904320q_1^2q_2q_3 + 463651200q_1^3q_2q_3 + 2400q_1q_2^2q_3 - 1808640q_1^2q_2^2q_3 \\ & + 3360q_1q_2^3q_3 + 4320q_1^2q_3^2 + 307054080q_1^3q_3^2 + 4800q_1q_2q_3^2 - 3617280q_1^2q_2q_3^2 \\ & + 33600q_1q_2^2q_3^2 + 10080q_1q_2q_3^3 + \dots \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} \kappa_{123}[X_n] = & 1 + 1440q_1q_2q_3 - 452160q_1^2q_2q_3 + 154550400q_1^3q_2q_3 + 4800q_1q_2^2q_3 \\ & + 10080q_1q_2^3q_3 - 1808640q_1^2q_2^2q_3 + 4800q_1q_2q_3^2 - 1808640q_1^2q_2q_3^2 + 67200q_1q_2^2q_3^2 \\ & + 10080q_1q_2q_3^3 + \dots \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} \kappa_{133}[X_n] = & 480q_1q_3 + 565776q_1^2q_3 + 51175680q_1^3q_3 + 1910067120q_1^4q_3 + 1440q_1q_2q_3 \\ & - 452160q_1^2q_2q_3 + 154550400q_1^3q_2q_3 + 2400q_1q_2^2q_3 - 904320q_1^2q_2^2q_3 + 3360q_1q_2^3q_3 \\ & + 4320q_1^2q_3^2 + 204702720q_1^3q_3^2 + 9600q_1q_2q_3^2 - 3617280q_1^2q_2q_3^2 + 67200q_1q_2^2q_3^2 \\ & + 30240q_1q_2q_3^3 + \dots \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} \kappa_{222}[X_n] = & -2q_2 + 480q_1q_2 + 282888q_1^2q_2 + 17058560q_1^3q_2 + 477516780q_1^4q_2 - 2q_2^2 \\ & + 4320q_1^2q_2^2 + 136468480q_1^3q_2^2 - 2q_2^3 - 2q_2^4 - 2q_2^5 - 4q_2q_3 + 1440q_1q_2q_3 \\ & - 226080q_1^2q_2q_3 + 51516800q_1^3q_2q_3 - 48q_2^2q_3 + 19200q_1q_2^2q_3 - 3617280q_1^2q_2^2q_3 \\ & - 216q_2^3q_3 + 90720q_1q_2^3q_3 - 640q_2^4q_3 - 6q_2q_3^2 + 2400q_1q_2q_3^2 - 452160q_1^2q_2q_3^2 - 260q_2^2q_3^2 \\ & + 134400q_1q_2^2q_3^2 - 2970q_2^3q_3^2 - 8q_2q_3^3 + 3360q_1q_2q_3^3 - 880q_2^2q_3^3 - 10q_2q_3^4 + \dots \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned}\kappa_{223}[X_n] = & -4q_2q_3 + 1440q_1q_2q_3 - 226080q_1^2q_2q_3 + 51516800q_1^3q_2q_3 - 24q_2^2q_3 \\ & + 9600q_1q_2^2q_3 - 1808640q_1^2q_2^2q_3 - 72q_2^3q_3 + 30240q_1q_2^3q_3 - 160q_2^4q_3 - 12q_2q_3^2 \\ & + 4800q_1q_2q_3^2 - 904320q_1^2q_2q_3^2 - 260q_2^2q_3^2 + 134400q_1q_2^2q_3^2 - 1980q_2^3q_3^2 - 24q_2q_3^3 \\ & + 10080q_1q_2q_3^3 - 1320q_2^2q_3^3 - 40q_2q_3^4 + \dots\end{aligned}\quad (\text{B.11})$$

$$\begin{aligned}\kappa_{233}[X_n] = & -4q_2q_3 + 1440q_1q_2q_3 - 226080q_1^2q_2q_3 + 51516800q_1^3q_2q_3 - 12q_2^2q_3 \\ & + 4800q_1q_2^2q_3 - 904320q_1^2q_2^2q_3 - 24q_2^3q_3 + 10080q_1q_2^3q_3 - 40q_2^4q_3 - 24q_2q_3^2 \\ & + 9600q_1q_2q_3^2 - 1808640q_1^2q_2q_3^2 - 260q_2^2q_3^2 + 134400q_1q_2^2q_3^2 - 1320q_2^3q_3^2 - 72q_2q_3^3 \\ & + 30240q_1q_2q_3^3 - 1980q_2^2q_3^3 - 160q_2q_3^4 + \dots\end{aligned}\quad (\text{B.12})$$

$$\begin{aligned}\kappa_{333}[X_n] = & -2q_3 + 480q_1q_3 + 282888q_1^2q_3 + 17058560q_1^3q_3 + 477516780q_1^4q_3 - 4q_2q_3 \\ & + 1440q_1q_2q_3 - 226080q_1^2q_2q_3 + 51516800q_1^3q_2q_3 - 6q_2^2q_3 + 2400q_1q_2^2q_3 \\ & - 452160q_1^2q_2^2q_3 - 8q_2^3q_3 + 3360q_1q_2^3q_3 - 10q_2^4q_3 - 2q_3^2 + 4320q_1^2q_3^2 + 136468480q_1^3q_3^2 \\ & - 48q_2q_3^2 + 19200q_1q_2q_3^2 - 3617280q_1^2q_2q_3^2 - 260q_2^2q_3^2 + 134400q_1q_2^2q_3^2 - 880q_2^3q_3^2 \\ & - 2q_3^3 - 216q_2q_3^3 + 90720q_1q_2q_3^3 - 2970q_2^2q_3^3 - 2q_3^4 - 640q_2q_3^4 - 2q_3^5 + \dots\end{aligned}\quad (\text{B.13})$$

B.2 \mathbb{F}_1

$$l^1 = \begin{pmatrix} -6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad l^2 = \begin{pmatrix} 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \end{pmatrix} \quad l^3 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 \end{pmatrix} \quad (\text{B.14})$$

$$q_1 = qU, \quad q_2 = \frac{qT}{qU}, \quad q_3 = \frac{qS}{(qUqT)^{\frac{1}{2}}}. \quad (\text{B.15})$$

$$24b_1 = 92, \quad 24b_2 = 36, \quad 24b_3 = 24 \quad (\text{B.16})$$

$$\kappa_{111}^0[X_n] = 8, \quad \kappa_{112}^0[X_n] = 3, \quad \kappa_{122}^0[X_n] = 1, \quad \kappa_{113}^0[X_n] = 2, \quad \kappa_{123}^0[X_n] = 1. \quad (\text{B.17})$$

$$\begin{aligned}\kappa_{111}[X_n] = & 8 + 480q_1 + 4320q_1^2 + 13440q_1^3 + 35040q_1^4 + 60480q_1^5 + 480q_1q_2 \\ & + 2263104q_1^2q_2 + 460581120q_1^3q_2 + 30561073920q_1^4q_2 + 4320q_1^2q_2^2 + 460581120q_1^3q_2^2 \\ & + 252q_1q_3 + 41040q_1^2q_3 + 1478520q_1^3q_3 + 26873280q_1^4q_3 - 960q_1q_2q_3 \\ & + 945360q_1^2q_2q_3 + 5029579008q_1^3q_2q_3 - 1920q_1q_2^2q_3 + 2712960q_1^2q_2^2q_3 - 2880q_1q_2^3q_3 \\ & - 73764q_1^2q_3^2 - 18191520q_1^3q_3^2 - 82080q_1^2q_2q_3^2 + 2400q_1q_2^2q_3^2 + \dots\end{aligned}\quad (\text{B.18})$$

$$\begin{aligned}\kappa_{112}[X_n] = & 3 + 480q_1q_2 + 1131552q_1^2q_2 + 153527040q_1^3q_2 + 7640268480q_1^4q_2 \\ & + 4320q_1^2q_2^2 + 307054080q_1^3q_2^2 - 960q_1q_2q_3 + 472680q_1^2q_2q_3 + 1676526336q_1^3q_2q_3 \\ & - 3840q_1q_2^2q_3 + 2712960q_1^2q_2^2q_3 - 8640q_1q_2^3q_3 - 41040q_1^2q_2q_3^2 + 4800q_1q_2^2q_3^2 + \dots\end{aligned}\quad (\text{B.19})$$

$$\begin{aligned}\kappa_{113}[X_n] = & 2 + 252q_1q_3 + 20520q_1^2q_3 + 492840q_1^3q_3 + 6718320q_1^4q_3 - 960q_1q_2q_3 \\ & + 472680q_1^2q_2q_3 + 1676526336q_1^3q_2q_3 - 1920q_1q_2^2q_3 + 1356480q_1^2q_2^2q_3 - 2880q_1q_2^3q_3 \\ & - 73764q_1^2q_3^2 - 12127680q_1^3q_3^2 - 82080q_1^2q_2q_3^2 + 4800q_1q_2^2q_3^2 + \dots\end{aligned}\quad (\text{B.20})$$

$$\begin{aligned}\kappa_{122}[X_n] = & 1 + 480q_1q_2 + 565776q_1^2q_2 + 51175680q_1^3q_2 + 1910067120q_1^4q_2 \\ & + 4320q_1^2q_2^2 + 204702720q_1^3q_2^2 - 960q_1q_2q_3 + 236340q_1^2q_2q_3 + 558842112q_1^3q_2q_3\end{aligned}\quad (\text{B.21})$$

$$-7680q_1q_2^2q_3 + 2712960q_1^2q_2^2q_3 - 25920q_1q_2^3q_3 - 20520q_1^2q_2q_3^2 + 9600q_1q_2^2q_3^2 + \dots$$

$$\kappa_{123}[X_n] = 1 - 960q_1q_2q_3 + 236340q_1^2q_2q_3 + 558842112q_1^3q_2q_3 - 3840q_1q_2^2q_3 \quad (\text{B.22})$$

$$+ 1356480q_1^2q_2^2q_3 - 8640q_1q_2^3q_3 - 41040q_1^2q_2q_3^2 + 9600q_1q_2^2q_3^2 + \dots$$

$$\kappa_{133}[X_n] = 252q_1q_3 + 10260q_1^2q_3 + 164280q_1^3q_3 + 1679580q_1^4q_3 - 960q_1q_2q_3 \quad (\text{B.23})$$

$$+ 236340q_1^2q_2q_3 + 558842112q_1^3q_2q_3 - 1920q_1q_2^2q_3 + 678240q_1^2q_2^2q_3 - 2880q_1q_2^3q_3$$

$$- 73764q_1^2q_3^2 - 8085120q_1^3q_3^2 - 82080q_1^2q_2q_3^2 + 9600q_1q_2^2q_3^2 + \dots$$

$$\kappa_{222}[X_n] = -2q_2 + 480q_1q_2 + 282888q_1^2q_2 + 17058560q_1^3q_2 + 477516780q_1^4q_2 - 2q_2^2 \quad (\text{B.24})$$

$$+ 4320q_1^2q_2^2 + 136468480q_1^3q_2^2 - 2q_2^3 - 2q_2^4 - 2q_2^5 + 3q_2q_3 - 960q_1q_2q_3 + 118170q_1^2q_2q_3$$

$$+ 186280704q_1^3q_2q_3 + 40q_2^2q_3 - 15360q_1q_2^2q_3 + 2712960q_1^2q_2^2q_3 + 189q_2^3q_3$$

$$- 77760q_1q_2^3q_3 + 576q_2^4q_3 - 10260q_1^2q_2q_3^2 - 45q_2^2q_3^2 + 19200q_1q_2^2q_3^2 - 864q_2^3q_3^2 + \dots$$

$$\kappa_{223}[X_n] = 3q_2q_3 - 960q_1q_2q_3 + 118170q_1^2q_2q_3 + 186280704q_1^3q_2q_3 + 20q_2^2q_3 \quad (\text{B.25})$$

$$- 7680q_1q_2^2q_3 + 1356480q_1^2q_2^2q_3 + 63q_2^3q_3 - 25920q_1q_2^3q_3 + 144q_2^4q_3 - 20520q_1^2q_2q_3^2$$

$$- 45q_2^2q_3^2 + 19200q_1q_2^2q_3^2 - 576q_2^3q_3^2 + \dots$$

$$\kappa_{233}[X_n] = 3q_2q_3 - 960q_1q_2q_3 + 118170q_1^2q_2q_3 + 186280704q_1^3q_2q_3 + 10q_2^2q_3 \quad (\text{B.26})$$

$$- 3840q_1q_2^2q_3 + 678240q_1^2q_2^2q_3 + 21q_2^3q_3 - 8640q_1q_2^3q_3 + 36q_2^4q_3 - 41040q_1^2q_2q_3^2$$

$$- 45q_2^2q_3^2 + 19200q_1q_2^2q_3^2 - 384q_2^3q_3^2 + \dots$$

$$\kappa_{333}[X_n] = q_3 + 252q_1q_3 + 5130q_1^2q_3 + 54760q_1^3q_3 + 419895q_1^4q_3 + 3q_2q_3 - 960q_1q_2q_3 \quad (\text{B.27})$$

$$+ 118170q_1^2q_2q_3 + 186280704q_1^3q_2q_3 + 5q_2^2q_3 - 1920q_1q_2^2q_3 + 339120q_1^2q_2^2q_3 + 7q_2^3q_3$$

$$- 2880q_1q_2^3q_3 + 9q_2^4q_3 + q_3^2 - 73764q_1^2q_3^2 - 5390080q_1^3q_3^2 - 82080q_1^2q_2q_3^2 - 45q_2^2q_3^2$$

$$+ 19200q_1q_2^2q_3^2 - 256q_2^3q_3^2 + q_3^3 + q_3^4 + q_3^5 + \dots$$

B.3 \mathbb{F}_2

$$l^1 = \begin{pmatrix} -6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad l^2 = \begin{pmatrix} 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \end{pmatrix} \quad l^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 \end{pmatrix} \quad (\text{B.28})$$

$$q_1 = q_U, \quad q_2 = \frac{q_T}{q_U}, \quad q_3 = \frac{q_S}{q_T}. \quad (\text{B.29})$$

$$24b_1 = 92, \quad 24b_2 = 48, \quad 24b_3 = 24 \quad (\text{B.30})$$

$$\kappa_{111}^0[X_n] = 8, \quad \kappa_{112}^0[X_n] = 4, \quad \kappa_{122}^0[X_n] = 2, \quad \kappa_{113}^0[X_n] = 2, \quad \kappa_{123}^0[X_n] = 1. \quad (\text{B.31})$$

$$\kappa_{111}[X_n] = 8 + 480q_1 + 4320q_1^2 + 13440q_1^3 + 35040q_1^4 + 60480q_1^5 + 480q_1q_2 \quad (\text{B.32})$$

$$+ 2263104q_1^2q_2 + 460581120q_1^3q_2 + 30561073920q_1^4q_2 + 4320q_1^2q_2^2 + 460581120q_1^3q_2^2$$

$$+ 480q_1q_2q_3 + 2263104q_1^2q_2q_3 + 460581120q_1^3q_2q_3 + 1440q_1q_2^2q_3 - 1808640q_1^2q_2^2q_3$$

$$+ 2400q_1q_2^3q_3 + \dots$$

$$\kappa_{112}[X_n] = 4 + 480q_1q_2 + 1131552q_1^2q_2 + 153527040q_1^3q_2 + 7640268480q_1^4q_2 \quad (\text{B.33})$$

$$+ 153527040q_1^3q_2q_3 + 4320q_1^2q_2^2 + 307054080q_1^3q_2^2 + 480q_1q_2q_3 + 1131552q_1^2q_2q_3$$

$$\begin{aligned}
& + 2880q_1q_2^2q_3 - 1808640q_1^2q_2^2q_3 + 7200q_1q_2^3q_3 + \dots \\
\kappa_{113}[X_n] = & 2 + 480q_1q_2q_3 + 1131552q_1^2q_2q_3 + 153527040q_1^3q_2q_3 + 1440q_1q_2^2q_3 \\
& - 904320q_1^2q_2^2q_3 + 2400q_1q_2^3q_3 + \dots
\end{aligned} \tag{B.34}$$

$$\begin{aligned}
\kappa_{122}[X_n] = & 2 + 480q_1q_2 + 565776q_1^2q_2 + 51175680q_1^3q_2 + 1910067120q_1^4q_2 \\
& + 4320q_1^2q_2^2 + 204702720q_1^3q_2^2 + 480q_1q_2q_3 + 565776q_1^2q_2q_3 + 51175680q_1^3q_2q_3 \\
& + 5760q_1q_2^2q_3 - 1808640q_1^2q_2^2q_3 + 21600q_1q_2^3q_3 + \dots
\end{aligned} \tag{B.35}$$

$$\begin{aligned}
\kappa_{123}[X_n] = & 1 + 480q_1q_2q_3 + 565776q_1^2q_2q_3 + 51175680q_1^3q_2q_3 + 2880q_1q_2^2q_3 \\
& - 904320q_1^2q_2^2q_3 + 7200q_1q_2^3q_3 + \dots
\end{aligned} \tag{B.36}$$

$$\begin{aligned}
\kappa_{133}[X_n] = & 480q_1q_2q_3 + 565776q_1^2q_2q_3 + 51175680q_1^3q_2q_3 + 1440q_1q_2^2q_3 \\
& - 452160q_1^2q_2^2q_3 + 2400q_1q_2^3q_3 + \dots
\end{aligned} \tag{B.37}$$

$$\begin{aligned}
\kappa_{222}[X_n] = & -2q_2 + 480q_1q_2 + 282888q_1^2q_2 + 17058560q_1^3q_2 + 477516780q_1^4q_2 - 2q_2^2 \\
& + 4320q_1^2q_2^2 + 136468480q_1^3q_2^2 - 2q_2^3 - 2q_2^4 - 2q_2^5 - 2q_2q_3 + 480q_1q_2q_3 + 282888q_1^2q_2q_3 \\
& + 17058560q_1^3q_2q_3 - 32q_2^2q_3 + 11520q_1q_2^2q_3 - 1808640q_1^2q_2^2q_3 - 162q_2^3q_3 \\
& + 64800q_1q_2^3q_3 - 512q_2^4q_3 - 2q_2^2q_3^2 - 162q_2^3q_3^2 + \dots
\end{aligned} \tag{B.38}$$

$$\begin{aligned}
\kappa_{223}[X_n] = & -2q_2q_3 + 480q_1q_2q_3 + 282888q_1^2q_2q_3 + 17058560q_1^3q_2q_3 + 5760q_1q_2^2q_3 \\
& - 16q_2^2q_3 - 904320q_1^2q_2^2q_3 - 54q_2^3q_3 + 21600q_1q_2^3q_3 - 128q_2^4q_3 - 2q_2^2q_3^2 - 108q_2^3q_3^2 + \dots
\end{aligned} \tag{B.39}$$

$$\begin{aligned}
\kappa_{233}[X_n] = & -2q_2q_3 + 480q_1q_2q_3 + 282888q_1^2q_2q_3 + 17058560q_1^3q_2q_3 - 8q_2^2q_3 \\
& + 2880q_1q_2^2q_3 - 452160q_1^2q_2^2q_3 - 18q_2^3q_3 + 7200q_1q_2^3q_3 - 32q_2^4q_3 - 2q_2^2q_3^2 - 72q_2^3q_3^2 + \dots
\end{aligned} \tag{B.40}$$

$$\begin{aligned}
\kappa_{333}[X_n] = & -2q_2q_3 + 480q_1q_2q_3 + 282888q_1^2q_2q_3 + 17058560q_1^3q_2q_3 - 4q_2^2q_3 \\
& + 1440q_1q_2^2q_3 - 226080q_1^2q_2^2q_3 - 6q_2^3q_3 + 2400q_1q_2^3q_3 - 8q_2^4q_3 - 2q_2^2q_3^2 - 48q_2^3q_3^2 + \dots
\end{aligned} \tag{B.41}$$

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